

Pointwise Complexity, Local Time, and Quantum Field

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Abstract

We develop a pointwise complexity viewpoint for Gaussian processes that refines the classical “global-supremum” paradigm. In the setting of Gaussian free fields and random walk on graphs, the Ray–Knight (isomorphism) theorem links the random-walk local-time field to a shifted square of the GFF, allowing our viewpoint to upgrade the study of cover and blanket times of Ding, Lee, and Peres, 2012 on graphs from a single-scalar analysis to a genuinely field-level one. The ambient equivalence of pointwise dimension (Lemma 8 in Li and Xu, 2026) and the “uniform pointwise convergence” principle in Xu and Zeevi, 2025 are nontrivial ingredients in establishing this local-time upper-bound characterization.

Beyond its probabilistic interest, this pointwise perspective resonates with constructive QFT, where cutoff-uniform local upper bounds are used to certify integrability and stability of non-perturbative path-integral formulations, even in regimes where a global supremum is ill behaved or does not exist. It also matches modern AI practice, where random-walk-based methods (e.g., representation learning on graphs) typically leverage local exploration rather than exhaustive coverage. Overall, the manuscript highlights a unified narrative connecting random walks, GFF/QFT, majorizing measures, and pointwise complexity, and argues that field-level bounds provide a natural and potentially powerful refinement of classical global-supremum analyses.

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Big Picture Summary

I Random Walk and Local Time

- A *graph* is a discrete space: points (vertices) connected by links (edges). The *graph Laplacian* is the graph analogue of the differential operator $-\Delta$ from PDE. As a result, many PDE notions (harmonic functions, Poisson equations, heat flow) have graph versions. This is why the study of graph is often related to the study of structure/physics/PDE.
- A *random walk* on a graph is a stochastic process that moves between adjacent vertices, either uniformly at random (the *simple random walk*) or according to edge weights. It is the probabilistic analogue of diffusion/heat flow on a discrete space: its transition probabilities satisfy a graph heat equation.
- *Local time* records how long the random walk spends at each vertex. It gives a detailed occupation profile of the walk, not just where it ends.

II Gaussian Free Field and Quantum Field Theory

- The *Gaussian free field (GFF)* is a canonical Gaussian random function on the vertices. It is a special case of general Gaussian process where the covariance is determined by the Laplacian: its covariance is the (pinned/killed) inverse Laplacian, also called a *Green’s function*.
- In theoretical physics, the (Euclidean) *free scalar quantum field* is a Gaussian field whose two-point function is a Green’s function.

$$\mu(d\phi) \propto \mathcal{Z}^{-1} \exp\left(-\frac{1}{2} \int_{\Omega} \|\nabla\phi(x)\|^2 dx\right) \mathcal{D}\phi,$$

And the GFF is the discrete analogue of this free field: the Dirichlet energy $\int \|\nabla\phi\|^2$ is replaced by the graph energy

$$\sum_{(u,v) \in E} (\eta_u - \eta_v)^2.$$

Constructive (i.e. non-perturbative) quantum field theory (nLab contributors, 2026) seeks to replace heuristic path-integral formulations with explicit, rigorous mathematical objects—often drawing on probabilistic constructions such as random walks and their local times.

- Deep *isomorphism theorems* (Ray–Knight / Dynkin / Eisenbaum, etc.) (Dynkin, 1984) connect local times of Markov processes to (shifts of) squares of Gaussian fields:

$$\left(L_{\tau(t)}(v) + \frac{1}{2}\eta_v^2\right)_{v \in V} \stackrel{d}{=} \left(\frac{1}{2}(\eta_v + \sqrt{2t})^2\right)_{v \in V},$$

providing a powerful bridge between random walks and field theory.

III Cover Time and Global Supremum of GFF

- Cover time and blanket time quantify how long a random walk needs to *thoroughly explore* a graph. It is a fundamental object described as (Aldous, 1989; Wilf, 1989; Dembo et al., 2004)

”Any mathematician will want to know how long, on the average, it takes until each pixel is visited.”

- A remarkable theorem of Ding, Lee, and Peres, 2012 shows that these exploration times are governed, up to universal constants, by the geometry of a Gaussian free field (GFF) on the graph, and equivalently by Talagrand’s γ_2 functional from majorizing-measure theory. In particular, the cover time is comparable to $|E| \cdot (\mathbb{E} \max_{v \in V} \eta_v)^2$, where η is the GFF, and is also comparable to $\gamma_2(V, \sqrt{\kappa})^2$ where κ is the commute-time metric:

$$t_{\text{cov}}(G) \asymp |E| \cdot \left(\mathbb{E} \max_{v \in V} \eta_v\right)^2 \asymp \gamma_2(V, \sqrt{\kappa})^2.$$

This bridges *random exploration* (random walks), *Gaussian geometry* (suprema of Gaussian processes), and *multiscale coverings* (majorizing measures).

IV Pointwise Complexity of Fields and Local Integrability

- **From scalars to fields.** Classical cover-time bounds control a *single number* (e.g., $t_{\text{cov}}(G)$) via a *single* Gaussian scalar such as $\mathbb{E} \max_{x \in V} \eta_x$. In contrast, random walks generate an entire *field* of local times $\{L_t(x)\}_{x \in V}$, and the Ray–Knight identity couples this field (in distribution) to a *shifted square* of a Gaussian field. This makes pointwise control of η_x intrinsically valuable.
- **A pointwise multiscale complexity.** Majorizing-measure theory admits a pointwise refinement: given a prior μ on V and the canonical metric $d(x, y) = (\mathbb{E}|\eta_x - \eta_y|^2)^{1/2}$, define

$$\Phi_\mu(x) := \int_0^\infty \sqrt{\log \frac{1}{\mu(B_d(x, \varepsilon))}} d\varepsilon.$$

This assigns to each x a *local* complexity profile, separating global suprema into pointwise geometric contributions.

- **Pointwise upper bounds without global finiteness.** Combining Lemma 8 in Li and Xu, 2026 with a uniform pointwise-convergence principle for indexed Gaussian processes, one obtains high-probability bounds of the form

$$\forall x \in V : \quad \eta_x \leq C \Phi_\mu(x) + C' \sqrt{\log(1/\delta)},$$

for universal constants $C, C' > 0$. Conceptually, this yields *local* control at each x via $\Phi_\mu(x)$, without requiring a global bound on $\sup_{x \in V} \eta_x$.

- **From pointwise Gaussian control to blanket/cover/local-time control.** Via the Ray–Knight isomorphism (local time \leftrightarrow shifted GFF square), pointwise bounds on η_x translate into pointwise control of the local-time field. In particular:
 - *Blanket time* concerns ratios $L_t(u)/L_t(v)$; under the isomorphism these compare shifted squares $(\eta_u + \sqrt{2t})^2 / (\eta_v + \sqrt{2t})^2$, making uniform pointwise control directly relevant.
 - *Cover time* concerns whether $\min_x L_t(x) = 0$; under the isomorphism this is governed by whether some η_x is near $-\sqrt{2t}$, recovering the extremal (supremum) scale while retaining a field-level perspective.
 - *Local times themselves* form a random field; pointwise complexity naturally targets this object and can describe how the occupation landscape builds across space and scales, beyond merely the last-visited vertex.
- **A bridge to constructive QFT.** In constructive QFT, ultraviolet (and sometimes infrared) singularities make the formal path integral ill-defined; sharp *pointwise* (local) upper bounds are central to certifying integrability and stability uniformly in the cutoff. Since the isomorphism theorems identify local-time fields with (renormalized) quadratic Gaussian observables, pointwise complexity offers a natural language for local integrability and cutoff-uniform control.

- **Applications to AI and science (why local exploration matters).** Local times have broad applications in science (e.g., biological networks), engineering (e.g., robotics exploration), and computing (e.g., search algorithms). In particular, many random-walk-based methods do *not* aim to cover an entire graph; instead they deliberately sample *local neighborhoods*. A canonical example is NODE2VEC, where biased random walks generate local contexts for representation learning on graphs (Grover and Leskovec, 2016). This practical gap motivates pointwise complexity.

Part I

Random Walk and Local Time

1 Graphs: A Discrete Geometry

A *graph* $G = (V, E)$ consists of a set of vertices V and edges E connecting pairs of vertices. One can think of a graph as a network, or as a discretized space. A function on a graph is simply a map $f : V \rightarrow \mathbb{R}$, assigning a number to each vertex.

The Laplacian $-\Delta$ is a central operator in PDE. On a graph, its discrete analogue is the (*combinatorial*) *graph Laplacian*

$$L = D - A,$$

where A is the adjacency matrix and D is the diagonal degree matrix with $D_{uu} = \deg(u)$. Applied to a function $f : V \rightarrow \mathbb{R}$, we have

$$(Lf)(u) = \deg(u) f(u) - \sum_{v \sim u} f(v) = \sum_{v \sim u} (f(u) - f(v)).$$

Thus $Lf(u)$ measures how different $f(u)$ is from the values of its neighbors—the discrete analogue of “second derivatives” or “curvature.”

Because of this, many PDE notions have graph analogues:

- **Harmonic functions:** $Lf = 0$ (discrete mean-value property).
- **Poisson equation:** $Lf = g$.
- **Heat equation:** $\partial_t u = -Lu$ (see the next section).

2 Random Walk: Diffusion on a Graph

A *simple random walk* on G is a Markov chain (X_t) on V such that from a vertex v it moves to a uniformly chosen neighbor.

Let $G = (V, E)$ be a finite, connected graph. The *simple random walk* is a Markov chain $(X_t)_{t \geq 0}$ on the vertex set V defined by:

- Start at some vertex $X_0 \in V$.

- If $X_t = v$, then X_{t+1} is chosen uniformly at random among the neighbors of v :

$$\mathbb{P}(X_{t+1} = u \mid X_t = v) = \begin{cases} \frac{1}{\deg(v)}, & \text{if } (v, u) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently, the chain has transition matrix P with entries $P(v, u) = 1/\deg(v)$ whenever u is adjacent to v .

Because the graph is finite and connected, this Markov chain is irreducible; if the graph is also non-bipartite, it is aperiodic and converges to a unique stationary distribution

$$\pi(v) = \frac{\deg(v)}{2|E|}.$$

In continuous time, the random walk is a natural model for diffusion on a discrete space. Let $p_t(u, v) = \mathbb{P}_u(X_t = v)$ denote the transition probabilities. In continuous time, they satisfy a graph heat equation:

$$\frac{d}{dt}p_t = -Lp_t, \quad \text{so} \quad p_t = e^{-tL}p_0.$$

Thus the random walk is the probabilistic counterpart of heat flow on the graph.

3 Local Time: How Much the Walk Visits Each Vertex

The random walk path records where the walker goes; *local time* records how long it stays at each vertex. For a continuous-time walk,

$$L_t(v) = \int_0^t \mathbf{1}\{X_s = v\} ds.$$

The vector $\{L_t(v)\}_{v \in V}$ gives an occupation profile over the graph. It is useful for studying exhaustive exploration (e.g., visiting all vertices) and fairness (e.g., how evenly the walk samples vertices).

A closely related object is the *Green's function* (in a setting where the integral is finite, e.g. after killing/pinning):

$$G(u, v) = \int_0^\infty \mathbb{P}_u(X_t = v) dt.$$

Intuitively, $G(u, v)$ is the expected total time spent at v when starting from u , and one has the identity

$$G(u, v) = \mathbb{E}_u[L_\infty(v)]$$

whenever $L_\infty(v)$ is finite (for example, for a killed process).

Part II

Gaussian Free Field and Quantum Field Theory

4 Gaussian Free Field: The Canonical Gaussian Field on a Graph

The *Gaussian free field (GFF)* assigns a Gaussian random variable η_v to each vertex v . It is not an arbitrary Gaussian field: it is the one naturally determined by the graph Laplacian.

Because the Laplacian L has a zero mode (constants), one typically *pins* the field at one vertex v_0 by setting $\eta_{v_0} = 0$. Let $L^{(v_0)}$ denote the Laplacian with the row and column of v_0 removed. Then the pinned GFF is the centered Gaussian vector with covariance

$$\text{Cov}(\eta) = (L^{(v_0)})^{-1}.$$

The matrix $(L^{(v_0)})^{-1}$ is exactly the (pinned/killed) Green's function. Thus, the GFF is the Gaussian field whose two-point correlations are given by the Green's function of the Laplacian.

5 Local time \leftrightarrow GFF: an explicit isomorphism theorem (Ray–Knight)

We now state a concrete identity that exemplifies the “isomorphism theorems” linking local times of Markov processes to (shifts of) squares of Gaussian fields.

Setup. Let $G = (V, E)$ be a finite, connected, undirected graph. Consider the continuous-time simple random walk $(X_t)_{t \geq 0}$ with generator $-L$, where L is the (combinatorial) graph Laplacian. Fix a vertex $v_0 \in V$ and define the local time

$$L_t(v) := \int_0^t \mathbf{1}\{X_s = v\} ds.$$

Let

$$\tau(t) := \inf\{s \geq 0 : L_s(v_0) \geq t\}$$

be the inverse local time at v_0 .

Let $\eta = \{\eta_v\}_{v \in V}$ be the *pinned* Gaussian free field with $\eta_{v_0} = 0$ and covariance given by the (killed/pinned) Green's function, equivalently

$$\text{Cov}(\eta) = (L^{(v_0)})^{-1},$$

where $L^{(v_0)}$ is the Laplacian with the row/column of v_0 removed (so it is invertible).

Theorem 1 (Second Ray–Knight isomorphism on a finite graph). *With the above definitions, the following identity holds in distribution:*

$$\left(L_{\tau(t)}(v) + \frac{1}{2}\eta_v^2\right)_{v \in V} \stackrel{d}{=} \left(\frac{1}{2}(\eta_v + \sqrt{2t})^2\right)_{v \in V}, \quad (1)$$

where $\eta_{v_0} = 0$ on both sides (and hence $L_{\tau(t)}(v_0) = t$). Equivalently, for each $v \in V$,

$$L_{\tau(t)}(v) \stackrel{d}{=} \frac{1}{2}(\eta_v + \sqrt{2t})^2 - \frac{1}{2}\eta_v^2 = t + \sqrt{2t}\eta_v.$$

Interpretation. Theorem 1 makes the slogan “*local time* \leftrightarrow *field squared*” precise: after stopping the walk at an inverse-local-time clock, the entire local-time profile $\{L_{\tau(t)}(v)\}_{v \in V}$ can be represented (in distribution) in terms of a shifted square of the pinned GFF. In the continuum (and in quantum field theory), the analogous statements involve Wick squares $:\phi^2:$ in place of the naive pointwise square, reflecting the renormalization required to define quadratic observables.

6 Why the Gaussian Free Field (GFF) is tied to (constructive) quantum field theory

One-sentence summary. The Gaussian free field (GFF) is the Euclidean version of the *free* (i.e., non-interacting) scalar quantum field, and is the most fundamental and mathematically tractable object in quantum field theory.

6.1 (a) Free field = Gaussian field; covariance = Green’s function (propagator)

In quantum field theory, the simplest field theory is the *free scalar field*. After passing to the Euclidean formulation, its (formal) path-integral measure takes the form

$$\mu(d\phi) \propto \mathcal{Z}^{-1} \exp\left(-\frac{1}{2} \int_{\Omega} \|\nabla\phi(x)\|^2 dx\right) \mathcal{D}\phi, \quad (2)$$

where the exponent is the (massless) action functional. Formally, (2) is a *Gaussian measure* on fields: the random field ϕ is Gaussian.

A Gaussian field is determined by its covariance (two-point function). In the free case, the covariance kernel $G(\cdot, \cdot)$ satisfies the Green’s function equation

$$(-\Delta)G(\cdot, y) = \delta_y, \quad (3)$$

and in field-theory language

$$\mathbb{E}[\phi(x)\phi(y)] = G(x, y), \quad (4)$$

so G is the *propagator*.

On a graph $G = (V, E)$, the GFF is the exact discrete analogue of this construction:

- The Dirichlet energy $\int \|\nabla\phi\|^2$ is replaced by the graph energy

$$\sum_{(u,v)\in E} (\eta_u - \eta_v)^2.$$

- The continuous Laplacian $-\Delta$ is replaced by the graph Laplacian L .
- The covariance is given by the inverse (or pseudo-inverse) of L , i.e., a graph Green's function.

Thus, the GFF is the discrete Euclidean free field.

6.2 (b) Why this matters for constructive quantum field theory

A central goal of *constructive* quantum field theory is to replace heuristic path-integral expressions (like $\mathcal{D}\phi$ in (2)) with explicit, rigorous mathematical objects (probability measures and random distributions). The free field is the most successful and canonical example:

- On a finite graph (finite-dimensional), the GFF is a bona fide Gaussian vector and is fully rigorous.
- In the continuum limit, the GFF becomes a well-defined generalized Gaussian random field (a distribution-valued object), again constructed rigorously.
- In two dimensions, the GFF serves as a foundational building block for many interacting theories (e.g., Liouville theory, sine-Gordon, and Coulomb-gas/CFT constructions), often obtained by reweighting or exponentiating the GFF (which requires renormalization).

In this sense, the GFF is both a fundamental object of quantum field theory and a base measure from which more complex fields are constructed.

6.3 (c) Why Wick squares appear

In the continuum, the free field ϕ is typically *not* a pointwise-defined function; it is a generalized random distribution. Consequently, $\phi(x)^2$ is not well-defined and diverges. To make quadratic observables (e.g., “energy density” or “field squared”) rigorous, one uses *Wick renormalization* (normal ordering):

$$:\phi^2:(x) = \lim_{\varepsilon\rightarrow 0} \left(\phi_\varepsilon(x)^2 - \mathbb{E}[\phi_\varepsilon(x)^2] \right), \quad (5)$$

where ϕ_ε is a smoothed version of ϕ . The subtraction term $\mathbb{E}[\phi_\varepsilon(x)^2]$ is precisely the (regularized) diagonal of the Green's function and captures the local divergence of $G(x, x)$. Wick renormalization is therefore indispensable for making “field squared” observables well-defined in continuum field theory.

Part III

Cover Time and Global Supremum of GFF

7 Cover time and blanket time

Let $G = (V, E)$ be a finite connected graph, and let $(X_t)_{t \geq 0}$ be the simple random walk on G . Write τ_{cov} for the first time at which every vertex has been visited, and let \mathbb{E}_v denote expectation for the walk started at $v \in V$. The (expected) cover time is

$$t_{\text{cov}}(G) := \max_{v \in V} \mathbb{E}_v[\tau_{\text{cov}}]. \quad (6)$$

Blanket time. Let $\pi(v) = \deg(v)/(2|E|)$ be the stationary distribution. Informally, a blanket time requires that the walk has not only visited every vertex, but has accumulated *roughly stationary proportions* of visits across all vertices. A strong δ -blanket time (one convenient definition) asks that the normalized occupation counts $N_u(t)/\pi(u)$ are within a multiplicative factor of δ for all pairs $u, v \in V$. One then defines

$$t_{\text{bl}}(G, \delta) := \max_{v \in V} \mathbb{E}_v[\tau_{\text{bl}}(\delta)]. \quad (7)$$

7.1 Majorizing measures and the γ_2 functional

Let (X, d) be a finite metric space. Talagrand's γ_2 functional is defined via admissible sequences of multiscale partitions $\{A_k\}_{k \geq 0}$:

$$\gamma_2(X, d) := \inf_{\{A_k\}} \sup_{x \in X} \sum_{k \geq 0} 2^{k/2} \text{diam}(A_k(x)), \quad (8)$$

where the infimum ranges over admissible partition sequences (increasingly fine, with controlled cardinalities).

For a centered Gaussian process $\{\eta_i\}_{i \in I}$, its canonical metric is

$$d(i, j) := (\mathbb{E}|\eta_i - \eta_j|^2)^{1/2}.$$

The majorizing-measures theorem states that

$$\mathbb{E} \sup_{i \in I} \eta_i \asymp \gamma_2(I, d). \quad (9)$$

8 Cover/blanket time $\leftrightarrow \gamma_2$ and the GFF

Let $H(u, v)$ be the expected hitting time from u to v , and define the commute-time metric

$$\kappa(u, v) := H(u, v) + H(v, u).$$

A classical identity gives $\kappa(u, v) = 2|E| R_{\text{eff}}(u, v)$, where R_{eff} is the effective resistance.

Theorem 2 (Cover/blanket times and majorizing measures). *For any connected graph $G = (V, E)$ and any fixed $0 < \delta < 1$,*

$$t_{\text{cov}}(G) \asymp \gamma_2(V, \sqrt{\kappa})^2 = |E| \cdot \gamma_2(V, \sqrt{R_{\text{eff}}})^2 \asymp_{\delta} t_{\text{bl}}(G, \delta). \quad (10)$$

Gaussian free field viewpoint. Let $\eta = \{\eta_v\}_{v \in V}$ be the (pinned) Gaussian free field (GFF) on G , characterized by $\mathbb{E}(\eta_u - \eta_v)^2 = R_{\text{eff}}(u, v)$. Then combining (9) with Theorem 2 yields

$$t_{\text{cov}}(G) \asymp |E| \cdot \left(\mathbb{E} \max_{v \in V} \eta_v \right)^2. \quad (11)$$

8.1 Isomorphism theorem: local time \leftrightarrow shifted GFF square

To discuss local times, it is often convenient to work in continuous time. Define the (normalized) local time at vertex x up to time t by

$$L_t(x) := \frac{1}{c_x} \int_0^t \mathbf{1}\{X_s = x\} ds, \quad c_x = \deg(x),$$

and define the inverse local time at a fixed root v_0 by

$$\tau(t) := \inf\{s : L_s(v_0) > t\}.$$

Theorem 1 shows the following. Fix $v_0 \in V$. Let $\eta = \{\eta_x\}_{x \in V}$ be a centered Gaussian process with covariance $\Gamma_{v_0}(x, y) = \mathbb{E}_x[L_{T_0}(y)]$, where T_0 is the hitting time of v_0 . Then for any $t > 0$, under the product measure of the walk and η ,

$$\left\{ L_{\tau(t)}(x) + \frac{1}{2}\eta_x^2 : x \in V \right\} \stackrel{d}{=} \left\{ \frac{1}{2}(\eta_x + \sqrt{2t})^2 : x \in V \right\}. \quad (12)$$

In particular, η is (a version of) the GFF on G , and (12) makes precise the slogan “*local time equals a shifted square of the GFF, in distribution.*”

Part IV

Pointwise Complexity of Fields and Local Integrability

9 Why pointwise (field-level) bounds matter

Single-value vs field-level control. Bounds such as (11) control a *single scalar* (the cover time) via a *single scalar* Gaussian quantity (the supremum $\max_v \eta_v$). However, Theorem 1 shows that the random walk produces an entire *field* $\{L_{\tau(t)}(x)\}_{x \in V}$, and this field is coupled (in distribution) to the shifted square field $\{(\eta_x + \sqrt{2t})^2\}_{x \in V}$. Therefore, if one can control η_x *pointwise across x* , then one can control the local-time field pointwise as well.

A pointwise majorizing-measure functional. Majorizing-measure theory can be expressed not only as a bound on $\mathbb{E} \sup_x \eta_x$, but also via a *pointwise multiscale complexity* (for a probability measure μ on V):

$$\Phi_\mu(x) := \int_0^\infty \sqrt{\log \frac{1}{\mu(B_d(x, \varepsilon))}} d\varepsilon, \quad B_d(x, \varepsilon) = \{y : d(x, y) \leq \varepsilon\},$$

where $d(x, y) = (\mathbb{E}|\eta_x - \eta_y|^2)^{1/2}$ is the canonical metric. A Fernique–Talagrand type inequality yields

$$\mathbb{E} \sup_{x \in V} \eta_x \lesssim \inf_\mu \sup_{x \in V} \Phi_\mu(x),$$

which separates the *global* supremum into *pointwise* geometric complexity profiles.

By Lemma 8 (Ambient Equivalence of Pointwise Dimension) in Li and Xu, 2026 and a uniform pointwise-convergence principle (Xu and Zeevi, 2025) for indexed Gaussian processes (rather than empirical process), we obtain the following high-probability bound: for any prior μ on V , with probability at least $1 - \delta$,

$$\forall x \in V : \quad \eta_x \leq C \Phi_\mu(x) + C' \sqrt{\log(1/\delta)}, \quad (13)$$

where $\Phi_\mu(x)$ denotes the pointwise Fernique–Talagrand integral (or its equivalent pointwise complexity functional) evaluated at x , and $C, C' > 0$ are universal constants.

This bound is conceptually significant: it provides *local* control of the Gaussian field at each x via a pointwise multiscale complexity, without requiring a *global* finiteness condition such as bounding $\sup_{x \in V} \eta_x$ (which is the quantity that typically appears in cover-time analyses).

From pointwise Gaussian control to blanket/cover/local-time control.

- **Blanket time:** By definition, blanket events compare ratios $L_t(u)/L_t(v)$ across u, v . Via (12), these ratios become comparable to ratios of shifted squares $(\eta_u + \sqrt{2t})^2 / (\eta_v + \sqrt{2t})^2$, so controlling η_x (or increments $\eta_u - \eta_v$) *uniformly over all vertices* yields control of blanket times.
- **Cover time:** Covering corresponds to the event $\exists x : L_t(x) = 0$, i.e., the minimum local time is zero. Under (12), this is governed by whether some η_x is close to $-\sqrt{2t}$; hence the extremal geometry of η drives the cover time scale, leading back to (11).
- **Local times (the full field):** Unlike $t_{\text{cov}}(G)$, the collection $\{L_{\tau(t)}(x)\}_{x \in V}$ is a random field. Pointwise bounds on η_x (or on the pointwise complexity $\Phi_\mu(x)$) suggest a route to field-level statements about local times: not merely when the last vertex is hit, but how the occupation profile builds up across the graph and across scales.

10 Why pointwise upper bounds matter in constructive quantum field theory

A central difficulty in quantum field theory (QFT) is that the *formal* path integral, as typically written, does not define a well-posed probability measure. When one attempts to

interpret it non-perturbatively as an actual distribution (or as an oscillatory integral in the real-time formulation), divergences arise, most notably in the ultraviolet (UV, small-scale) regime and sometimes also in the infrared (IR, large-scale) regime.

In *constructive* QFT, the objective is to replace such heuristic expressions with explicit, rigorous mathematical objects. This requires establishing sharp *upper bounds*—including moment bounds, exponential integrability estimates, and stability bounds—that are uniform in the cutoff (e.g., a lattice spacing, mollification scale, or finite-volume truncation). Such bounds enable one to:

1. prove that the interacting weight (e.g., $e^{-\int V(\phi)}$) is integrable, so that the partition function is finite;
2. ensure that the theory is stable (i.e., there are no runaway negative directions); and
3. remove cutoffs and take limits (continuum and/or infinite-volume) in a controlled manner.

In summary, the main obstruction to making the path integral rigorous is that it typically diverges when interpreted non-perturbatively as a measure, due to ultraviolet (and sometimes infrared) singularities. Establishing sharp upper bounds—stability and integrability estimates that are uniform in the cutoff—identifies which interactions admit well-defined calculations and controlled limits, thereby turning the formal path integral into a genuine probabilistic object.

11 Applications to AI and science

Cover time has broad applications in exhaustive search and exploration, ranging from immune-system cells chasing pathogens and animals harvesting resources to robotic exploration (e.g., cleaning or demining) and the design of search algorithms (Chupeau, Bénichou, and Voituriez, 2015). A closely related notion at the level of local times is the *first-passage* (hitting) time, which more broadly models transport and search phenomena in complex environments, including disordered media, neuronal firing dynamics, disease spreading, and target-search processes (Condamin et al., 2007).

A practical bottleneck in bringing random-walk theory to modern AI is that many random-walk-based algorithms do not aim to explore the entire graph (as cover time does). Instead, they deliberately sample *local neighborhoods*. A canonical example is the celebrated NODE2VEC algorithm, where (biased) random walks are used to generate local contexts for representation learning on graphs (Grover and Leskovec, 2016). This gap motivates our focus on *pointwise* complexity and *local-time* fields: understanding and controlling local exploration profiles—rather than only global exhaustive coverage—has substantial potential value for both theory and practice.

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